

Characterization of Fully Coupled FBSDE in Terms of Portfolio Optimization under Probability and Discounting Uncertainty

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ABSTRACT

We characterize a class of fully coupled forward backward stochastic differential equations in terms of optimal maximal sub-solutions of BSDEs. We present the application thereof in utility optimization with random endowment under probability and discounting uncertainty. We provide some explicit examples and show how to quantify the costs of incompleteness when using utility indifference pricing.

KEYWORDS: Fully Coupled FBSDE, Portfolio Optimization, Random Endowment, Probability and Discounting Uncertainty.

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1 Introduction

We study solutions to the following type of coupled forward backward stochastic differential systems

$$\begin{cases} X_t &= x + \int_0^t \hat{\pi}(X, Y, Z, V) \cdot d\hat{W} \\ Y_t &= F - \int_t^T g(\pi(X, Y, Z, V), X, Y, Z) ds - \int_t^T Z \cdot dW \\ U_t &= U_T + \frac{1}{2} \int_t^T (\hat{V}^2 - \tilde{V}^2) ds + \int_t^T V \cdot dW \\ U_T &= \int_0^T \left(\partial_y g(\pi(X, Y, Z, V), X, Y, Z) + \frac{\partial_z g(\pi(X, Y, Z, V), X, Y, Z)^2}{2} \right) ds \\ &\quad + \int_0^T \partial_z g(\pi(X, Y, Z, V), X, Y, Z) \cdot d\tilde{W} \end{cases} \quad (1.1)$$

where

- $W = (\hat{W}, \tilde{W})$ is a d dimensional Brownian motion whereby \hat{W} and \tilde{W} denote the first n and last $n - d$ components, respectively;
- g is a convex driver;
- F is a bounded terminal condition.

Systems of similar flavor often arise as result of stochastic optimization problems. Starting from this observation, our goal is to provide a characterization of such systems in terms of optimal control process π that maximizes the value Y_0 , maximal sub-solution of the decoupled forward backward stochastic differential equation

$$\begin{cases} X_t &= x + \int_0^t \hat{\pi} \cdot d\hat{W} \\ Y_s &\leq Y_t - \int_s^t g(\pi, X, Y, Z) du - \int_s^t Z \cdot dW, \quad 0 \leq s \leq t \leq T \\ Y_T &\leq F \end{cases} \quad (1.2)$$

Maximal sub-solutions of backward stochastic differential equations have been introduced and studied by Drapeau et al. [7], Heyne et al. [14]. Maximal sub-solutions can be understood as an extension of backward stochastic differential equations, where equality is dropped in favor of inequality allowing weaker conditions for the driver g . It allows to achieve existence, uniqueness and comparison theorem without growth assumptions on the driver as well as weaker integrability condition on the forward process and terminal condition. To stress the relation between maximal sub-solutions and solutions of backward stochastic differential equations, maximal sub-solutions can be characterized as maximal viscosity sub-solutions in the Markovian case, see [6]. It also turns out that they are particularly adequate for optimization problem in terms of convexity or duality among others, see [8, 13]. Since existence and uniqueness holds for a larger class of backward stochastic differential equations, we can characterize systems (1.1) in a larger framework than the classical BSDE theory.

Literature Discussion Utility optimization problems in continuous time are popular topics in finance. Karatzas et al. [18] considered the optimization of the expected discounted utility of both consumption and terminal wealth in the complete market where they obtained an optimal consumption and wealth processes explicitly. Using duality methods, Cvitanic et al. [5] characterized the problem of utility maximization from terminal wealth of an agent with a random endowment process in semi-martingale model for incomplete markets. Backward stochastic differential equations, introduced in the seminal paper by Pardoux and Peng [21] in the Lipschitz case and Kobylanski [20] for the quadratic one, have revealed to be central in stating and solving problems in finance, see El Karoui et al. [10]. Duffie and Epstein [9] defined the concept of recursive utility by means of backward stochastic differential equations, generalized in Chen and Epstein [4] and Quenez and Lazrak [24]. Utility optimization characterization in that context has been treated in El Karoui et al. [11] in terms of a forward backward system of stochastic differential equations. Using a martingale argumentation, Hu et al. [17] characterized utility maximization by means of quadratic backward stochastic differential equations for small traders on incomplete financial markets with closed constraints. Following this line with a general utility function, Horst et al. [16] characterized the optimal strategy via a fully-coupled forward backward stochastic differential equation. With a similar characterization, Santacrose and Trivellato [27] considered the problem with a terminal random liability when the underlying asset price process is a continuous semi-martingale. Bordigoni et al. [2] studied a stochastic control problem arising in utility maximization under probability model uncertainty given by the relative entropy, see also Schied [28], Anis et al. [1]. Backward stochastic differential equations, can be viewed themselves as generalized utility operators – so called g -expectations introduced by Peng [22] – which are related to risk measures, Rosazza Gianin [25], Peng [23], Rosazza Gianin [26]. Also, maximal sub-solutions of concave backward stochastic differential equations are also nonlinear expectations. In this respect, Heyne et al. [15] consider utility optimization in that framework, providing existence of optimal strategy using duality

methods as well as existence of gradients. However they do not provide a characterization of the optimal solution to which this work is dedicated to.

Discussion of the results The existence and uniqueness of minimal sub-solution – in this work, maximal super-solutions – is the subject of [7, 14]. It depends foremost on the integrability of the positive part of the terminal condition F , admissibility conditions on the local martingale part, and the properties of the generator – linear boundedness from below, lower semi-continuity, convexity in z and monotonicity in y or joint convexity in (y, z) . In this paper though, we have to change the conditions on the drivers in terms of boundedness from below as well as the admissibility condition which is more adequate for the optimization problem we are seeking at. We provide existence and uniqueness of maximal sub-solutions under these new admissibility conditions. The key point throughout is to keep track of the sub-martingale property for the corresponding control process under discounting and probability measure changes. With this result at hand, we can address the characterization in terms of optimization of maximal subsolutions of the forward backward stochastic differential equation. It turns out, that an auxiliary backward stochastic differential equation is necessary in order to specify the gradient of the solution. We apply the result by considering utility optimization in a financial context with explicit solutions in some examples. We finish how this can be used to compare the additional costs of completeness when using utility indifference pricing.

Outline of the paper The paper is structured as follows. Section 2 set up the notations and present the main results of this work as well as their translation in a financial context. Section 3 is dedicated to the proofs of the main results. Section 4 present the financial application of this result, the study of examples in the complete and incomplete market as well as the cost of incompleteness in indifference pricing.

2 Presentation of the Main Results

2.1 Notations

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a filtrated probability space, where the filtration (\mathcal{F}_t) is generated by a d -dimensional Brownian motion W and fulfills the usual conditions. We further assume that $\mathcal{F} = \mathcal{F}_T$. Throughout, we split this d dimensional Brownian motion into two parts $W = (\hat{W}, \tilde{W})$ with $\hat{W} = (W^1, \dots, W^n)$ and $\tilde{W} = (W^{n+1}, \dots, W^d)$ where $1 \leq n \leq d$. We denote by L^0 the set of \mathcal{F}_T -measurable random variables identified in the P -almost sure sense. Every inequality between random variables is to be understood in the almost sure sense. Furthermore as in the introduction, to keep the notational burden as minimal as possible, we do not write the index in t and ω for the integrands unless necessary. We furthermore generically use the short writing $\int \cdot$ for the process $t \mapsto \int_0^t \cdot$ and say that a process X is integrable if X_t is integrable for every $0 \leq t \leq T$. We use the notations

- $xy = (x_1 y_1, \dots, x_d y_d)$ and $x/y = (x_1/y_1, \dots, x_d/y_d)$ for x and y in \mathbb{R}^d
- $x \cdot y = \sum x_k y_k$, $x^2 = x \cdot x$ and $|x| = \sqrt{x \cdot x}$ for x and y in \mathbb{R}^d .
- for $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$

$$x \cdot A \cdot y := \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

- g^* denotes the convex conjugate of a function and denote by $\partial g = (\partial_r g, \partial_x g, \partial_y g, \partial_z g)$ the sub-gradients of g .¹
- \mathcal{S} the set of càdlàg adapted processes.
- \mathcal{L} the set of \mathbb{R}^d -valued predictable processes Z such that $\int Z \cdot dW$ is a local martingale.²
- \mathcal{H} the set of local martingales $\int Z \cdot dW$ for $Z \in \mathcal{L}$.
- \mathcal{L}^p the set of those Z in \mathcal{L} such that $\|Z\|_{\mathcal{L}^p} := E[(\int_0^T Z^2 dt)^{p/2}]^{1/p} < \infty$, $1 \leq p < \infty$.
- \mathcal{H}^p the set martingales $\int Z \cdot dW$ for $Z \in \mathcal{L}^p$.
- bmo the set of those Z in \mathcal{L} such that $\int Z \cdot dW$ has bounded mean oscillations.³
- \mathcal{D} the set of those uniformly bounded $b \in \mathcal{L}$.
- M^c the stochastic exponential of c , that is $M^c = \exp(-\int c \cdot dW - \frac{1}{2} \int c^2 dt)$.
- D^b the stochastic discounting of b , that is $D^b = \exp(-\int b ds)$.
- $M^{bc} = D^b M^c = \exp(-\int (b + c^2/2) dt - \int c \cdot dW)$.
- For $Z \in \mathcal{L}$ we denote by $Z = (\hat{Z}, \tilde{Z})$, the corresponding decomposition along \hat{W} and \tilde{W} , the same for the space \mathcal{H} , \mathcal{H}^p , bmo and BMO .

2.2 Maximal Sub-Solutions of FBSDEs

As mentioned in the introduction, we want to characterize the system (1.1) as the optimal maximal sub-solution of a Forward Backward Stochastic Differential Equation. A function $g : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \rightarrow (-\infty, \infty]$ is called a *generator* if it is jointly measurable. A generator is said to satisfy condition (STD) if

(STD) g is jointly lower semi-continuous, jointly convex with non-empty interior and non-empty gradient everywhere on its domain.

For any strategy $\hat{\pi}$ in $\hat{\mathcal{L}}$ and *terminal condition* $F \in L^0$, we call a pair (Y, Z) where $Y \in \mathcal{S}$ and $Z \in \mathcal{L}$ a sub-solution of the forward backward stochastic differential equation if

$$\begin{cases} X_t = x + \int_0^t \hat{\pi} \cdot d\hat{W}, & 0 \leq t \leq T \\ Y_s \leq Y_t - \int_s^t g(\hat{\pi}, X, Y, Z) du - \int_s^t Z \cdot dW, & 0 \leq s \leq t \leq T \\ Y_T \leq F \end{cases} \quad (2.1)$$

The processes Y and Z are called the *value* and *control* processes, respectively. Sub-solutions are naturally not unique. Indeed, note that (Y, Z) is a sub-solution if and only if there exists an adapted càdlàg increasing process K with $K_0 = 0$ such that

$$Y_t = F - \int_t^T g(\hat{\pi}, X, Y, Z) ds - (K_T - K_t) - \int_t^T Z \cdot dW,$$

¹ By classical measurable selection argument if the gradients are not empty, for adapted processes $\hat{\pi}$, X , Y and Z we denote generically $\partial g(\hat{\pi}, X, Y, Z)$ a measurable selection of the gradient.

² That is $\int_0^T Z^2 dt < \infty$.

³ That is, $\sup_\tau \|E[\int_\tau^T Z \cdot dW | \mathcal{F}_\tau]\|_\infty < \infty$ where τ runs over all stopping times.

which is given by

$$K_t = Y_t - Y_0 - \int_0^t g(\hat{\pi}, X, Y, Z) ds - \int_0^t Z \cdot dW. \quad (2.2)$$

Subject of studies in [7, 12–14], existence and uniqueness of a maximal one depend foremost on the integrability of the positive part of F , admissibility conditions on the local martingale part, and the properties of the generator – linear boundedness from below, lower semi-continuity, convexity in z and monotonicity in y or joint convexity in (y, z) . In this paper though, we removed the condition on the driver in terms of boundedness from below due to the optimization problem we are looking at. In order to guarantee the existence and uniqueness of a maximal sub-solution, we need an adequate admissibility condition.

Definition 2.1. A sub-solution (Y, Z) to (2.1) is called *admissible* if for every predictable processes a, b , and c in \mathcal{L} such that

- c is in bmo , b is in \mathcal{D} , and $\int aM^{bc}dt$ is integrable;
- $g(r, x, y, z) \geq by + cz - a$;

the value process $M^{bc}Y^+$ is integrable.

Given a strategy $\hat{\pi}$, we denote by

$$\mathcal{A}^{\hat{\pi}} := \{(Y, Z) \in \mathcal{S} \times \mathcal{L} : (Y, Z) \text{ is an admissible sub-solution of (2.1)}\}$$

A pair (Y, Z) is called a *maximal sub-solution* of the FBSDE (2.1) if it is a sub-solution and if for every other sub-solution (\bar{Y}, \bar{Z}) it holds $\bar{Y} \leq Y$. Our first result concerns the existence and uniqueness of maximal sub-solution and a super-martingale property of the control process.

Theorem 2.2. Let $\hat{\pi}$ be in \hat{bmo} , $F \in L^\infty$ and g be a generator satisfying

- (STD);
- $g(\hat{\pi}, X, y, z) \geq by + cz - a$ for some processes c in bmo and b in \mathcal{D} and a such that $\int aM^{bc}dt$ is integrable.

If $\mathcal{A}^{\hat{\pi}}$ is non-empty, then there exists a unique maximal sub-solution (Y, Z) for which holds

$$Y_t = \text{essinf} \{ \bar{Y}_t : (\bar{Y}, \bar{Z}) \in \mathcal{A}^{\hat{\pi}} \}, \quad 0 \leq t \leq T$$

and $\int (M^{bc}Z - M^{bc}Yc) \cdot dW$ is a sub-martingale.

2.3 Characterization of the coupled FBSDE System

Theorem 2.2 allows us to formulate our main result, namely, the characterization in terms of optimal maximal sub-solution of the system (1.1). Indeed, we get an operator $\hat{\pi} \mapsto \mathcal{E}(\hat{\pi})$ that associate a control $\hat{\pi}$ in \hat{bmo} to the corresponding value process of the maximal sub-solution with the convention that if $\mathcal{A}^{\hat{\pi}}$ is empty, then we set $\mathcal{E}(\hat{\pi}) = -\infty$. We are interested into the following optimization problem

$$\mathcal{E}_0(\hat{\pi}) \xrightarrow{\hat{\pi} \in \hat{bmo}} \max$$

We say that $\hat{\pi}^*$ in \hat{bmo} is an *optimal strategy*, if $\mathcal{E}_0(\hat{\pi}) \leq \mathcal{E}_0(\hat{\pi}^*)$ for every $\hat{\pi} \in \mathcal{L}^1$. To simplify notations in the presentation of our main result we denote by

$$b := \partial_y g(\hat{\pi}, X, Y, Z) \quad , \quad c := (\hat{c}, \bar{c}) = (\partial_z g, \partial_z g)(\hat{\pi}, X, Y, Z) \quad \text{and} \quad a := g^*(\bar{c}, b, b, c) - \bar{c} \cdot \hat{\pi} - bX$$

for a given $\hat{\pi}$, $X = x + \int \hat{\pi} \cdot d\hat{W}$, Y and Z .

Theorem 2.3. Let $\hat{\pi} := \hat{\pi}(x, y, z, v)$ be such that

$$\partial_y g(\hat{\pi}, x, y, z) = \partial_x g(\hat{\pi}, x, y, z) \quad \text{and} \quad \partial_z g(\hat{\pi}, x, y, z) = \partial_r g(\hat{\pi}, x, y, z) = \hat{v} \quad (2.3)$$

Suppose that the fully coupled forward backward system of stochastic differential equations

$$\begin{cases} X_t &= x + \int_0^t \hat{\pi}(X, Y, Z, V) \cdot d\hat{W} \\ Y_t &= F - \int_t^T g(\hat{\pi}(X, Y, Z, V), X, Y, Z) ds - \int_s^T Z \cdot dW \\ U_t &= U_T + \frac{1}{2} \int_t^T (\hat{V}^2 - \tilde{V}^2) ds + \int_t^T V \cdot dW \\ U_T &= \int_0^T \left(\partial_y g(\hat{\pi}(X, Y, Z, V), X, Y, Z) + \frac{\partial_z g(\hat{\pi}(X, Y, Z, V), X, Y, Z)^2}{2} \right) ds \\ &\quad + \int_0^T \partial_z g(\hat{\pi}(X, Y, Z, V), X, Y, Z) \cdot d\tilde{W} \end{cases} \quad (2.4)$$

admits a solution (X^*, Y^*, Z^*, U, V) such that

- $\hat{\pi}^* := \hat{\pi}(X^*, Y^*, Z^*, V)$ is in $b\hat{m}o$;
- b is in \mathcal{D} and \tilde{c} is in $b\tilde{m}o$
- $\int M^{bc} g^*(\hat{c}, b, b, c) dt$ is integrable.

Then, $\hat{\pi}^*$ is an optimal strategy and (Y^*, Z^*) is solution of the “linear”⁴ backward stochastic differential equation

$$Y_t^* = F - \int_t^T [bY^* + c \cdot Z^* - a] ds - \int_t^T Z^* \cdot dW.$$

2.4 Financial Applications

The system of forward backward stochastic differential equations arises in particular in financial applications. Since the driver g can take extended real values, it allows for various convex constraints. We present here after in two different cases how to derive g from classical utility optimization problems. We consider a financial market consisting of one bond with interest rate 0 and a n -dimensional stock price \hat{S} that evolves according to

$$\frac{d\hat{S}}{\hat{S}} = \hat{\mu} dt + \hat{\sigma} \cdot d\hat{W}, \quad \text{and} \quad \hat{S}_0 \in \mathbb{R}_{++}^n$$

where $\hat{\mu}$ is a \mathbb{R}^n -valued uniformly bounded drift process, and $\hat{\sigma}$ is a $n \times n$ volatility matrix process. For simplicity, we assume that $\hat{\sigma}$ is reversible such that the market price of risk process $\hat{\theta} := \hat{\sigma}^{-1} \cdot \hat{\mu}$ is in $b\hat{m}o$. In the case where $n = d$, then the market is complete. Given a n -dimensional trading strategy $\hat{\eta}$, the corresponding wealth process with initial wealth x satisfies

$$X_t = x + \int_0^t \hat{\eta} \cdot \frac{d\hat{S}}{\hat{S}} = x + \int_0^t \hat{\eta} \cdot \hat{\sigma} \cdot (\hat{\theta} ds + d\hat{W}) = x + \int_0^t \hat{\eta} \cdot \hat{\sigma} \cdot d\hat{W}^{\hat{\theta}},$$

⁴Naturally, the coefficients a , b and c depends a fortiori on $\hat{\pi}, X, Y, Z$, but are characterized as the gradients estimated at the value of the optimal solution.

where $\hat{W}^\theta = \hat{W} + \int \hat{\theta} ds$ is a n -dimensional Brownian motion under $P^\theta = M_T^\theta P$. We adopt the notation $W^\theta = (\hat{W}^\theta, \tilde{W})$ which is a d -dimensional Brownian motion under P^θ . To remove the volatility factor, we generically set $\hat{\pi} = \hat{\eta} \cdot \hat{\theta}$.

The motivation for the optimization problem is as follows. Throughout, we fix a driver $h: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-n} \rightarrow [0, \infty]$ such that $(y, z) = (y, \hat{z}, \tilde{z}) \mapsto h(y, z) = h(y, \hat{z}, \tilde{z})$ is jointly convex and lower semi-continuous. We denote by $\mathcal{A}(F)$ the set of sub-solutions (\bar{Y}, \bar{Z}) in $\mathcal{S} \times \mathcal{L}$ of the backward stochastic differential equation

$$\begin{cases} \bar{Y}_s & \leq \bar{Y}_t - \int_s^t h(\bar{Y}, \bar{Z}) ds - \int_s^t \bar{Z} \cdot dW_s, & 0 \leq s \leq t \leq T \\ \bar{Y}_T & \leq F \end{cases}$$

where $\int \bar{Z} dW$ is a sub-martingale. According to [7], it follows that if $F^+ \in L^1$ and $\mathcal{A}(F)$ is non-empty, then there exists a unique maximal sub-solution. We denote by $\mathcal{E}(F)$ the value process of this maximal sub-solution, and convene that if $\mathcal{A}(F)$ is empty, then we set $\mathcal{E}(F) \equiv -\infty$. It follows from [7, 8], that $F \mapsto \mathcal{E}_0(F)$ is a concave and $\sigma(L^1, L^\infty)$ -lower semi-continuous functional. Furthermore, if g is increasing in y , then it satisfies the sub-cash additivity property, namely $\mathcal{E}_0(F - m) \geq \mathcal{E}_0(F) - m$ for every $m \geq 0$. In that case, this sub-cash additive monetary utility functional admits a dual representation in terms of the convex conjugate h^* of h

$$\mathcal{E}_0(F) = \inf_{\{b \in \mathcal{D}, c \in bmo: M_T^c \in L^\infty\}} \left\{ E \left[M_T^{bc} F + \int_0^T M^{bc} h^*(b, c) dt \right] \right\}, \quad F \in \mathcal{H}^1$$

allowing an interpretation of $F \mapsto \mathcal{E}_0(F)$ as a utility functional excerpting *probability* as well as *discounting uncertainty*. We are interested in a utility maximization problem with random endowment F in L^∞ , of the form

$$\mathcal{E}_0(\hat{\pi}) := \mathcal{E}_0 \left(F + x + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta \right) \xrightarrow[\text{trading strategies } \hat{\pi}]{\text{Over all admissible}} \max$$

and assume that $\hat{\pi}$ is in $b\hat{m}o$.

Fixing $\hat{\pi}$ in $b\hat{m}o = bmo$ and (\bar{Y}, \bar{Z}) in $\mathcal{A}(F + x + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta)$, the variable change $Y = \bar{Y} - X$ and $Z = \bar{Z} - \hat{\pi}$ is equivalent to the following

$$\begin{cases} X_t & := x + \int_0^t \hat{\pi} \cdot d\hat{W}^\theta \\ Y_s & \leq Y_t - \int_s^t g(\hat{\pi}, X, Y, Z) du - \int_s^t Z \cdot dW^\theta, & 0 \leq s \leq t \leq T \\ Y_T & \leq F \end{cases}$$

where

$$g(r, x, y, z) = g(r, x, y, \hat{z}, \tilde{z}) = h(y + x, \hat{z} + r, \tilde{z}) - (\hat{z} + r) \cdot \hat{\theta}$$

According to Lemma 3.1, the admissibility assumption is then easier since (\bar{Y}, \bar{Z}) is in $\mathcal{A}(F + X_T)$ if and only if $(Y, Z) = (\bar{Y} - X, \bar{Z} - \hat{\pi})$ is an admissible sub-solution in $\mathcal{A}^{\hat{\pi}}$ for the driver g . Furthermore, since $Y_0 = \bar{Y}^0 - X_0 = \bar{Y}^0 - x$, it follows that if Y is greater than every other maximal sub-solution for all admissible $\hat{\pi}$ then, so is \bar{Y} .

Complete Financial Market We consider a complete market, that is $n = d$. In that case since $\hat{\theta}$ and \hat{W}^θ are d -dimensional, we adopt the notation $\theta := \hat{\theta}$ and $W^\theta := \hat{W}^\theta$. It holds $g(r, x, y, z) =$

$h(y+x, z+r) - (z+r) \cdot \theta$. Applying the sufficient conditions with this special form of g yields automatically

$$\partial_y g = \partial_x g \quad \text{as well as} \quad \partial_z g = \partial_r g$$

and therefore the point wise optimal $\pi(x, y, z, v) = \pi(x+y, z, v)$ is characterized as

$$\partial_z h(y+x, z+\pi(x, y, z, v)) - \theta = v$$

and the corresponding Forward Backward Stochastic Differential Equation reads as

$$\begin{cases} \pi^* &= \pi(Y^* + X^*, Z^*, V) \\ V &= \partial_z h(Y^* + X^*, Z^* + \pi^*) - \theta \\ X_t^* &= x + \int_0^t \pi^* dW^\theta \\ Y_t^* &= F - \int_t^T (h(Y^* + X^*, Z^* + \pi^*) ds - (Z^* + \pi^*) \cdot \hat{\theta}) ds - \int_t^T Z^* dW^\theta \\ U_t &= U_T - \frac{1}{2} \int_t^T V^2 ds + \int_t^T V \cdot dW^\theta \\ U_T &= \int_0^T \partial_y h(Y^* + X^*, Z^* + \pi^*) ds \end{cases} \quad (2.5)$$

Incomplete Financial Market In the incomplete financial market case, that is, $1 \leq n < d$ it holds $g(r, x, y, z) = g(r, x, y, \hat{z}, \bar{z}) = h(y+x, \hat{z}+r, \bar{z}) - (\hat{z}+r) \cdot \hat{\theta}$. Here also we have

$$\partial_y g = \partial_x g \quad \text{as well as} \quad \partial_z g = \partial_r g$$

But the point wise optimal $\hat{\pi}(x, y, z, v) = \hat{\pi}(x+y, z, v)$ is characterized in that case as

$$\partial_z h(y+x, \hat{z}+\hat{\pi}(x, y, z, v), \bar{z}) - \hat{\theta} = \hat{v}$$

and the corresponding Forward Backward Stochastic Differential Equation reads as

$$\begin{cases} \hat{\pi}^* &= \hat{\pi}(Y^* + X^*, Z^*, V) \\ \hat{V} &= \partial_z h(Y^* + X^*, \hat{Z}^* + \hat{\pi}^*, \bar{Z}^*) - \hat{\theta} \\ \tilde{c} &= \partial_{\bar{z}} h(Y^* + X^*, \hat{Z}^* + \hat{\pi}^*, \bar{Z}^*) \\ X_t^* &= x + \int_0^t \hat{\pi}^* d\hat{W}^{\hat{\theta}} \\ Y_t^* &= F - \int_t^T (h(Y^* + X^*, \hat{Z}^* + \hat{\pi}^*, \bar{Z}^*) - (\hat{Z}^* + \hat{\pi}^*) \cdot \hat{\theta}) ds - \int_t^T Z^* \cdot dW^{\hat{\theta}} \\ U_t &= U_T + \frac{1}{2} \int_t^T (\hat{V}^2 - \tilde{V}^2) ds + \int_t^T V \cdot dW^{\hat{\theta}} \\ U_T &= \int_0^T \left(\partial_y h(Y^* + X^*, \hat{Z}^* + \hat{\pi}^*, \bar{Z}^*) + \frac{\tilde{c}^2}{2} \right) ds + \int_0^T \tilde{c} \cdot d\tilde{W}^{\hat{\theta}} \end{cases} \quad (2.6)$$

In Section 4 we provide examples in the complete and incomplete market.

Utility Indifference Pricing: The Cost of Incompleteness Since $x \mapsto \mathcal{E}_0(F + x + \int \hat{\pi}^* \cdot d\hat{W}^{\hat{\theta}})$ is an increasing function of x , we can look at the classical utility indifference pricing, that is

$$\mathcal{E}_0(F) = \mathcal{E}_0\left(F + x + \int_0^T \hat{\pi}^* \cdot d\hat{W}\right)$$

which compare the capital x required so that in terms of utility, one is indifferent between hedging or not hedging the claim. In other terms, the gain in terms of having access to a financial market.

Since our functional is only upper semi-continuous, we proceed as follows. To distinguish between the complete and incomplete market, we denote by π and $\hat{\pi}$ the set of d -dimensional and n -dimensional strategies, and define

$$\begin{aligned} x^* &= \inf \left\{ x \in \mathbb{R} : \sup_{\pi \in bmo} \mathcal{E}_0 \left(F + x + \int_0^T \pi \cdot dW^\theta \right) > \mathcal{E}_0(F) \right\} \\ y^* &= \inf \left\{ y \in \mathbb{R} : \sup_{\hat{\pi} \in \hat{bmo}} \mathcal{E}_0 \left(F + y + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta \right) > \mathcal{E}_0(F) \right\} \end{aligned}$$

which represents the utility indifference prices for F in the complete and incomplete case, respectively. Intuitively, the price is higher in the incomplete case, which is the subject of the following proposition.

Proposition 2.4. *For every F in L^∞ such that $\mathcal{E}_0(F) < \infty$ it holds that $x^* \leq y^*$.*

Proof. We denote by I and J the set of those x and y in \mathbb{R} such that $\mathcal{E}_0(F + x + \int_0^T \pi \cdot dW^\theta) > \mathcal{E}_0(F)$ and $\mathcal{E}_0(F + y + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta) > \mathcal{E}_0(F)$ for some π in bmo and $\hat{\pi}$ in \hat{bmo} , respectively. Since \hat{bmo} is a subset of bmo , it follows that $I \subseteq J$ showing that $x^* = \inf I \leq \inf J = y^*$. \square

In Section 4 based on the examples in the complete and incomplete market, we present explicit solutions for x^* and y^* and discuss the impact of incompleteness and the driver on y^* with respect to x^* .

3 Proof of the Main Results

It turns out that one of the key point to characterize the fully coupled system of forward backward stochastic differential equations (2.4), is the sub-martingale property of the control process. Before starting with the proof of Theorem 2.2, let us address a first result that shows how the admissibility condition of sub-solutions implies the sub-martingale property. We first fix $\hat{\pi}$ in \hat{bmo} .

Lemma 3.1. *For every processes a, b and c in \mathcal{L} satisfying the conditions of the admissibility requirements, it follows that $\int (M^{bc}Z - M^{bc}Yc) \cdot dW$ is a sub-martingale for every (Y, Z) in $\mathcal{A}^{\hat{\pi}}$.*

Proof. For $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$, the variable change $\tilde{Y} = M^{bc}Y + \int aM^{bc}dt$ and $\tilde{Z} = M^{bc}Z - M^{bc}Yc$ yields by Ito's formula⁵ that (\tilde{Y}, \tilde{Z}) satisfies

$$\begin{cases} \tilde{Y}_s \leq \tilde{Y}_t - \int_s^t \tilde{g}(\hat{\pi}, X, \tilde{Y}, \tilde{Z})du - \int_s^t \tilde{Z} \cdot dW \\ \tilde{Y}_T \leq M_T^{bc}F + \int_0^T aM^{bc}dt \end{cases}$$

⁵Recall that K in equation (2.2) is increasing.

where

$$\begin{aligned} \tilde{g}(r, x, \tilde{y}, \tilde{z}) := M^{bc} g \left(r, x, \frac{\tilde{y} - \int a M^{bc} dt}{M^{bc}}, \frac{\tilde{z} + \left(\int a M^{bc} dt \right) c - \tilde{y} c}{M^{bc}} \right) \\ - b \left(\tilde{y} - \int a M^{bc} dt \right) - c \cdot \left(\tilde{z} + \left(\int a M^{bc} dt \right) c - \tilde{y} c \right) + a M^{bc} \end{aligned} \quad (3.1)$$

is by assumption positive. It follows that

$$\int_0^t \tilde{Z} \cdot dW \leq \tilde{Y}_t - \tilde{Y}_0 - \int_0^t \tilde{g}(\hat{\pi}, X, \tilde{Y}, \tilde{Z}) du \leq \tilde{Y}_t - \tilde{Y}_0$$

for every t . By assumption, it follows that $(\int \tilde{Z} \cdot dW)^+$ is integrable and therefore, by classical argumentation,⁶ $\int \tilde{Z} \cdot dW$ is a sub-martingale. \square

Using this Lemma 3.1, we can apply the results in [7, 8] to prove Theorem 2.2.

Proof (Theorem 2.2). Processing to the same variable change as in Lemma 3.1, $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$ if and only if $\tilde{Y} = M^{bc} Y + \int_0^\cdot M_t^{bc} a_t dt$ and $\tilde{Z} = M^{bc} Z - c M^{bc} Y$ is a sub-solution of

$$\begin{cases} \tilde{Y}_s \leq \tilde{Y}_t - \int_s^t \tilde{g}(\hat{\pi}, X, \tilde{Y}, \tilde{Z}) du - \int_s^t \tilde{Z} \cdot dW, & 0 \leq s \leq t \leq T \\ \tilde{Y}_T \leq M_T^{bc} F + \int_0^T a M^{bc} dt \end{cases}$$

with $\int \tilde{Z} \cdot dW$ being a sub-martingale and \tilde{g} as in (3.1) is a positive driver. Since $M_T^{bc} F + \int_0^T a M^{bc} dt$ is integrable, it follows that the assumptions of [7, Theorem 4.1 p. 3984] are fulfilled, and therefore, there exists a unique maximal sub-solution (\tilde{Y}, \tilde{Z}) . Since M^{bc} is strictly positive, we deduce that there exists a unique maximal sub-solution $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$. \square

Before addressing the proof of the main Theorem 2.3, let us first note the following. As a consequence of Burkholder-Davis-Gundis inequality, it is known that $\|\cdot\|_{\mathcal{L}^p}$ and $\|\cdot\|_{\mathcal{H}^p}$ where $\|\int Z \cdot dW\|_{\mathcal{H}^p} = E[(\sup_{0 \leq t \leq T} |\int_0^t Z \cdot dW|^p)]^{1/p}$ are equivalent. It is also known that BMO is the dual space of \mathcal{H}^1 , the pairing of which is given by $E[\langle \int Z \cdot dW, \int c \cdot dW \rangle_T] = E[\int_0^T Z \cdot c dt]$ for Z in \mathcal{L}^1 and c in bmo and it holds

$$E[\langle \int Z \cdot dW, \int c \cdot dW \rangle_T] \leq C \left\| \int c \cdot dW \right\|_{BMO} \left\| \int Z \cdot dW \right\|_{\mathcal{H}^1}.$$

Since M^{bc} is a uniformly integrable process for b in \mathcal{D} and c in bmo , it follows that $E[M_T^{bc} | \mathcal{F}] - E[M_T^{bc}]$ is in \mathcal{H}^1 . For every $\hat{\pi}$ in \hat{bmo} , denoting by $\int H \cdot dW = E[M_T^{bc} | \mathcal{F}] - E[M_T^{bc}]$, on the one hand, it holds

$$\left\langle \int H \cdot dW, \int \hat{\pi} \cdot d\hat{W} \right\rangle = \int \hat{H} \cdot \hat{\pi} dt = \int H \cdot dW \int \hat{\pi} \cdot d\hat{W} - \int \left(\int H \cdot dW \right) \hat{\pi} \cdot d\hat{W} + \left(\int \hat{\pi} \cdot d\hat{W} \right) H \cdot dW \quad (3.2)$$

⁶Taking a localizing sequence of stopping times τ^n such that $(\int \tilde{Z} \cdot dW)^{\tau^n}$ is a martingale, it follows from Fatou's lemma that $\infty > E[\int_0^t \tilde{Z} \cdot dW] \geq \limsup E[\int_0^{t \wedge \tau^n} \tilde{Z} \cdot dW] = 0$ showing that $\int \tilde{Z} \cdot dW$ is integrable for every $0 \leq t \leq T$. For $0 \leq s \leq t \leq T$ and $A \in \mathcal{F}_s$, the same argumentation yields $E[1_A \int_s^t \tilde{Z} \cdot dW] \geq \limsup E[1_A \int_s^{t \wedge \tau^n} \tilde{Z} \cdot dW] = 0$ showing that $\int \tilde{Z} \cdot dW$ is a sub-martingale.

On the other hand,

$$M^{bc} \int \hat{\pi} \cdot d\hat{W} = \int M^{bc} \hat{\pi} \cdot d\hat{W} - \int M^{bc} \left(\int \hat{\pi} \cdot d\hat{W} \right) c \cdot dW - \int M^{bc} \left(\hat{c} \cdot \hat{\pi} + b \int \hat{\pi} \cdot d\hat{W} \right) dt \quad (3.3)$$

In the case where the local martingales in both equations (3.2) and (3.2) are true martingales, we have in particular the following relation ship between the dual pairing of \mathcal{H}^1 -BMO and the expression of M^{bc} as follows

$$E \left[\left\langle \int H \cdot dW, \int \hat{\pi} \cdot d\hat{W} \right\rangle_T \right] = E \left[M_T^{bc} \int_0^T \hat{\pi} \cdot d\hat{W} \right] = -E \left[\int_0^T M^{bc} \left(\hat{c} \cdot \hat{\pi} + b \int \hat{\pi} \cdot d\hat{W} \right) dt \right] \quad (3.4)$$

This relation in particular the last equality is central in the proof of the main Theorem 2.3. However, it is in general not true that the local martingale appearing resulting from the dual pairing between \mathcal{H}^1 and BMO are true martingales as stated in [19, Remark 2.5, Page 37]. Therefore the following two lemmas that guarantee Relation (3.4) as well as a characterization of M^{bc} in terms of quadratic backward stochastic differential equation.

Lemma 3.2. *Let b in \mathcal{D} and c in bmo . For $E[M_T^{bc}|\mathcal{F}] - E[M_T^{bc}] =: \int H \cdot dW$, and for every $\hat{\pi}$ in bmo , it holds that $M^{bc} \int \hat{\pi} \cdot d\hat{W}$ is integrable and for every $0 \leq t \leq T$, we have*

$$E \left[\left\langle \int H \cdot dW, \int \hat{\pi} \cdot d\hat{W} \right\rangle_t \right] = E \left[\left(\int_0^t H \cdot dW \right) \left(\int_0^t \hat{\pi} \cdot d\hat{W} \right) \right] \quad (3.5)$$

$$E \left[M_t^{bc} \int_0^t \hat{\pi} \cdot d\hat{W} \right] = -E \left[\int_0^t M^{bc} \left(\hat{c} \cdot \hat{\pi} + b \int \hat{\pi} \cdot d\hat{W} \right) ds \right] \quad (3.6)$$

Proof. For every $1 < p < \infty$, we denote by q the conjugate of p . Note first that since c is in bmo , the reverse Hölder inequality yields for some $1 < r < \infty$, that $E[(M_t^c)^r] \leq C_r(M_0^c)^r = C_r$ for some constant C_r independent of t . We denote by \tilde{r} the conjugate of r .

First, since D^b is uniformly bounded from above by a constant C and M^{bc} is positive, it follows that $-E[M_T^{bc}] \leq \int H \cdot dW \leq CM^c - E[M_T^{bc}]$ as well as $M^{bc} \leq CM^c$. Hence,

$$E \left[\left\| \int_0^t H \cdot dW \right\| \left\| \int_0^t \hat{\pi} \cdot d\hat{W} \right\| \right] \leq CE[(M_t^c)^r]^{1/r} \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^{\tilde{r}}} + (C - E[M_T^{bc}]) \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^{\tilde{r}}}$$

From $\hat{\pi}$ in $b\hat{m}o$, it follows that $\left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^{\tilde{r}}} < \infty$. From c is in bmo , by the reverse Hölder inequality, it follows that $E[(M_t^c)^r] \leq C_r(M_0^c)^r = C_r$. Hence $(\int H \cdot dW)(\int \hat{\pi} \cdot d\hat{W})$ is integrable.

Second, since $\int H \cdot dW$ is in \mathcal{H}^1 and $\hat{\pi}$ in $b\hat{m}o$, $\langle \int H \cdot dW, \int \hat{\pi} \cdot d\hat{W} \rangle$ is integrable. From Equation (3.2), it follows that $\int (\hat{\pi} \int H \cdot dW + H \int \hat{\pi} \cdot d\hat{W}) \cdot dW$ is a martingale. Hence equality (3.5).

Third, for the same reasons, $M^{bc} \int \hat{\pi} \cdot d\hat{W}$ is integrable. Furthermore

$$\begin{aligned} E \left[\int_0^t |M^{bc} \hat{c} \cdot \hat{\pi}| ds \right] &\leq CE \left[\int_0^t |M^c \hat{c} \cdot \hat{\pi}| ds \right] = CE \left[\int_0^t d \left\langle M^c, \int \hat{\pi} \cdot d\hat{W} \right\rangle \right] \\ &\leq C \|M^c\|_{\mathcal{H}^1} \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{BMO} < \infty \end{aligned}$$

Using Hölder and Fubini-Tonelli we have

$$\begin{aligned} E \left[\int_0^t |M^{bc} b \int \hat{\pi} \cdot d\hat{W}| ds \right] &\leq CE \left[\int_0^T |M^c \int \hat{\pi} \cdot d\hat{W}| ds \right] \leq CE \left[\sup_{0 \leq t \leq T} \left| \int_0^t \hat{\pi} \cdot d\hat{W} \right| \int_0^T (M^c) ds \right] \\ &\leq C \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^{\tilde{r}}} E \left[\int_0^T (M^c)^r ds \right]^{1/r} = C \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^{\tilde{r}}} \left[\int_0^T E[(M_t^c)^r] dt \right]^{1/r} \end{aligned}$$

Using reverse Hölder inequality $E[(M_t^c)^r] \leq C_r(M_0^c)^r = C_r$, yields

$$E \left[\int_0^t \left| M^{bc} b \int \hat{\pi} \cdot d\hat{W} \right| ds \right] \leq C(C_r T)^{1/r} \left\| \int \hat{\pi} \cdot d\hat{W} \right\|_{\mathcal{H}^r} < \infty$$

By Equation (3.3), it follows that $\int M^{bc} \hat{\pi} \cdot d\hat{W} - \int M^{bc} \left(\int \hat{\pi} \cdot d\hat{W} \right) c \cdot dW$ is a martingale and taking expectation, yields equality (3.6). \square

Lemma 3.3. *Let $b \in \mathcal{D}$ and $\tilde{c} \in b\tilde{m}o$. The backward stochastic differential equation*

$$\begin{cases} U_t &= U_T + \frac{1}{2} \int_t^T (\hat{V}^2 - \tilde{V}^2) ds + \int_t^T V \cdot dW \\ U_T &= \int_0^T \left(b + \frac{\tilde{c}^2}{2} \right) dt + \int_0^T \tilde{c} d\tilde{W} \end{cases}$$

admits a unique solution (U, V) with V in bmo .

In this case, if we define $c = (\hat{V}, \tilde{c})$ which is in bmo , it follows that b and c satisfy the assumptions of Lemma 3.2 and it holds

$$M_T^{bc} = \exp \left(- \int_0^T \frac{\delta^2}{2} dt - \int_0^T \delta \cdot d\tilde{W} - U_0 \right). \quad (3.7)$$

where $\delta = -\tilde{V}$ is in $b\tilde{m}o$.

Proof. According to Kobylanski [20], since $\int_0^T b ds$ is uniformly bounded, the following backward stochastic differential equation

$$Y_t = \int_0^T b ds + \int_t^T \left(\frac{\hat{Z}^2}{2} - \frac{\tilde{Z}^2}{2} \right) ds + \int_t^T Z \cdot dW^{\tilde{c}}$$

where $W^{\tilde{c}} = (\hat{W}, \tilde{W} + \int \tilde{c} ds)$ is a Brownian motion under $P^{\tilde{c}}$, admits a unique solution (Y, Z) where Y is uniformly bounded and Z is in $\mathcal{L}^2(W^{\tilde{c}})$. According to Briand and Elie [3, Proposition 2.1] it also holds that Z is in $bmo(W^{\tilde{c}})$ which is also bmo since \tilde{c} is bmo .⁷ The variable change $U = Y + \int \tilde{c}^2/2 dt + \int \tilde{c} \cdot d\tilde{W}$ and $V = (\hat{Z}, \tilde{Z} - \tilde{c})$ which is in bmo yields

$$\begin{aligned} U_t &= U_T + \int_t^T \left(\frac{\hat{V}^2}{2} + \tilde{c} \cdot \tilde{Z} - \frac{\tilde{Z}^2}{2} - \frac{\tilde{c}^2}{2} \right) ds + \int_t^T \hat{Z} \cdot d\hat{W} + \int_t^T (\tilde{Z} - \tilde{c}) \cdot d\tilde{W} \\ &= U_T + \frac{1}{2} \int_t^T (\hat{V}^2 - \tilde{V}^2) ds + \int_t^T V \cdot dW \end{aligned}$$

showing the first assertion. Defining now $c = (\hat{V}, \tilde{c})$, which is in bmo , it follows that

$$\begin{aligned} - \int_0^T \left(b - \frac{c^2}{2} \right) dt - \int_0^T c \cdot dW &= - \int_0^T \left(b - \frac{\tilde{c}^2}{2} \right) dt - \int_0^T \tilde{c} \cdot d\tilde{W} - \int_0^T \frac{\hat{V}^2}{2} dt - \int_0^T \hat{V} \cdot d\hat{W} \\ &= - \frac{1}{2} \int_0^T \tilde{V}^2 dt + \int_0^T \tilde{V} \cdot d\tilde{W} - U_0 = - \int_0^T \frac{\delta^2}{2} dt - \int_0^T \delta \cdot d\tilde{W} - U_0 \end{aligned}$$

where $\delta = -\tilde{V}$ is in $b\tilde{m}o$. Taking the exponential on both sides, yields (3.7). \square

⁷The bmo space is invariant under bmo measure change.

We are now in place to show the characterization of the fully coupled forward backward stochastic differential equation (2.4) of Theorem 2.3. Recall that we generically denote by $\partial g(r, x, y, z) = (\partial_r g, \partial_x g, \partial_y g, \partial_z g)$ some sub-gradient of g at (r, x, y, z) in the domain of g . Furthermore, the selection of gradient happens in a stochastic way and some classical measurable selection arguments allows to ensure measurability. Due to the conditions (2.3) it follows from classical convex analysis that for every (r, x, y, z) it holds

$$\begin{cases} g(r, x, y, z) \geq by + c \cdot z - (g^*(\hat{c}, b, b, c) - \hat{c} \cdot r - bx) = by + c \cdot z - a + \hat{c} \cdot (r - \hat{\pi}^*) + b(x - X^*) \\ g(\hat{\pi}^*, X^*, Y^*, Z^*) = bY^* + c \cdot Z^* - a \end{cases} \quad (3.8)$$

Proof (of Theorem 2.3). Let $\hat{\pi}^* := \hat{\pi}(X^*, Y^*, Z^*, V)$ and X^*, Y^*, Z^*, V be a solution of (2.4). On the first hand, since V is assumed to be bmo , it follows that we are in the setting of Lemma 3.3 so that, in particular, the processes b and c satisfy the conditions of Lemma 3.2. Hence, it follows that $\int (\hat{c} \cdot \hat{\pi} + b \int \hat{\pi} \cdot d\hat{W}) M^{bc} dt$ is integrable for every $\hat{\pi}$ in $b\hat{m}o$. Furthermore, denoting by $X = x + \int \hat{\pi} \cdot d\hat{W}$, by (3.8) it holds

$$g(\hat{\pi}, X, y, z) \geq by + c \cdot z - (g^*(\hat{c}, b, b, c) - \hat{c} \cdot \hat{\pi} - bX)$$

for every y, z and by assumption, $\int M^{bc} g^*(c, b, b, c) dt$ is integrable, it follows that

$$\int \left(g^*(\hat{c}, b, b, c) - \hat{c} \cdot \hat{\pi} - b \int \hat{\pi} \cdot d\hat{W} \right) M^{bc} dt,$$

is integrable. It holds in particular for $\hat{\pi}^*$ showing that $\int a M^{bc} dt$ is also integrable. Hence, for every $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$, it holds that $\int (M^{bc} Z - M^{bc} Y c) \cdot dW$ is a sub-martingale according to Lemma 3.1.

The fact that (Y^*, Z^*) is solution of the linear backward differential equation is a consequence of the equality $g(\hat{\pi}^*, X^*, Y^*, Z^*) = bY^* + cZ^* - a$. In that case, it follows in particular that $M^{bc} Z^* - M^{bc} Y^* c$ is the martingale representation of $M_T^{bc} F + \int_0^T a M^{bc} dt$ which is integrable since F is bounded and the fact that $\int_0^T a M^{bc} dt$ is integrable. Therefore $\int (M^{bc} Z^* - M^{bc} Y^* c) \cdot dW$ is a martingale

Let now $\hat{\pi}$ in $b\hat{m}o$, and $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$. For ease of notations, let $\Delta Y = Y - Y^*$, $\Delta Z = Z - Z^*$, $\Delta \hat{\pi} = \hat{\pi} - \hat{\pi}^*$, $\Delta X = X - X^* = \int \Delta \hat{\pi} \cdot d\hat{W}$. By means of Equation (3.8), it follows that

$$\begin{aligned} \Delta Y_s &\leq \Delta Y_t - \int_s^t [g(\hat{\pi}, X, Y, Z) - g(\hat{\pi}^*, X^*, Y^*, Z^*)] du - \int_s^t \Delta Z \cdot dW \\ &\leq \Delta Y_t - \int_s^t \left[\hat{c} \cdot \Delta \hat{\pi} + b \int \Delta \hat{\pi} \cdot d\hat{W} + b \Delta Y + c \cdot \Delta Z \right] du - \int_s^t \Delta Z \cdot dW \end{aligned}$$

By the change of variable $\tilde{Y} = M^{bc} \Delta Y - \int M^{bc} (\hat{c} \cdot \Delta \hat{\pi} + b \int \Delta \hat{\pi} \cdot d\hat{W}) dt$ and $\Delta \tilde{Z} = M^{bc} \Delta Z - M^{bc} \Delta Y c$, it follows that (\tilde{Y}, \tilde{Z}) satisfies

$$\tilde{Y}_t \leq - \int_0^T M^{bc} (\hat{c} \cdot \Delta \hat{\pi} + b \int \Delta \hat{\pi} \cdot d\hat{W}) ds - \int_t^T \tilde{Z} \cdot dW \leq -E \left[\int_t^T M^{bc} \left(\hat{c} \cdot \Delta \hat{\pi} + b \int \Delta \hat{\pi} \cdot d\hat{W} \right) ds \middle| \mathcal{F}_t \right]$$

since $\int \tilde{Z} \cdot dW$ is a sub-martingale as the difference between a sub-martingale and a martingale. However, c and b satisfying the condition of lemma 3.2 and 3.3, according to (3.7), it holds that

for $t = 0$ we have

$$\begin{aligned}\Delta Y_0 = \tilde{Y}_0 &\leq -E \left[\int_0^T M^{b,c} \left(\hat{c} \cdot \Delta \hat{\pi} + b \int \Delta \hat{\pi} \cdot d\hat{W} \right) dt \right] = E \left[M_T^{b,c} \int_0^T \Delta \hat{\pi} \cdot d\hat{W} \right] \\ &= E \left[\exp \left(- \int_0^T \frac{\delta^2}{2} dt - \int_0^T \delta \cdot d\tilde{W} - U_0 \right) \int_0^T \Delta \hat{\pi} \cdot d\hat{W} \right]\end{aligned}$$

with $\delta = -\tilde{V}$ which is in $b\tilde{m}o$. Since $\exp(-\int_0^T \frac{\delta^2}{2} dt - \int_0^T \delta \cdot d\tilde{W})$ in \tilde{H}^1 is orthogonal to $\int \hat{\pi} \cdot d\hat{W}$ in $B\hat{M}O$, it follows that the expectation on the right-hand side is equal to 0. Thus, $Y_0^* \geq Y_0$. \square

4 Financial Applications and Examples

In this Section in the context of the financial market presented in Section 2, we first address the question of admissibility. Then we present an explicit example in a complete market and another example stating the difference of price when maximizing utility in complete versus incomplete market.

Recall that we consider a financial market consisting of one bond with interest rate 0 and a n -dimensional stock price \hat{S} that evolves according to

$$\frac{d\hat{S}_t}{\hat{S}_t} = \hat{\mu} dt + \hat{\sigma} \cdot d\hat{W}, \quad \text{and} \quad S_0 \in \mathbb{R}_{++}^d$$

such that the market price of risk process $\hat{\theta} := \hat{\sigma}^{-1} \cdot \hat{\mu}$ is in $b\hat{m}o$. The wealth process is given by

$$X_t = x + \int_0^t \hat{\pi} \cdot d\hat{W}^{\hat{\theta}}$$

where $\hat{W}^{\hat{\theta}} = \hat{W} + \int \hat{\theta} dt$ is a n -dimensional Brownian motion under $P^{\hat{\theta}} = M_T^{\hat{\theta}} P$ and $\hat{\pi} = \hat{\eta} \cdot \hat{\sigma}$ is the trading strategy. Throughout this section, we consider all the notations under the measure $\hat{\theta}$ as well as $M^c = \exp(-\int c^2/2 dt - \int c \cdot dW^{\hat{\theta}})$ where $W^{\hat{\theta}} = (\hat{W}^{\hat{\theta}}, W)$ is a d dimensional Brownian motion under $P^{\hat{\theta}}$. We denote by $\mathcal{A}(F)$ the set of sub-solutions (Y, Z) in $\mathcal{S} \times \mathcal{L}$ of the backward stochastic differential equation

$$\begin{cases} Y_s &\leq Y_t - \int_s^t h(Y, Z) ds - \int_s^t Z \cdot dW_s, \quad 0 \leq s \leq t \leq T \\ Y_T &\leq F \end{cases}$$

where $\int Z dW$ is a sub-martingale for a driver $h: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-n} \rightarrow [0, \infty]$ such that $(y, z) = (y, \hat{z}, \tilde{z}) \mapsto h(y, z) = h(y, \hat{z}, \tilde{z})$ is jointly convex, lower semi-continuous. We denote by $\mathcal{E}_0(F)$ the value of the maximal sub-solution at 0. In the complete as well as the incomplete case, we search for an optimal strategy that maximize

$$\mathcal{E}_0 \left(F + x + \int_0^T \hat{\pi} \cdot d\hat{W}^{\hat{\theta}} \right) \xrightarrow[\text{strategies } \hat{\pi} \text{ in } b\hat{m}o]{\text{over all admissible}} \max$$

We showed that it is equivalent to maximize the maximal sub-solutions to

$$\begin{cases} X_t &:= x + \int_0^t \hat{\pi} \cdot d\hat{W}^{\hat{\theta}} \\ Y_s &\leq Y_t - \int_s^t g(\hat{\pi}, X, Y, Z) du - \int_s^t Z \cdot dW^{\hat{\theta}}, \quad 0 \leq s \leq t \leq T \\ Y_T &\leq F \end{cases}$$

where

$$g(r, x, y, z) = g(r, x, y, \hat{z}, \bar{z}) = h(y + x, \hat{z} + r, \bar{z}) - (\hat{z} + r) \cdot \hat{\theta}$$

The first question is however how admissibility carries over from one set to the other. In this particular case, the admissibility is equivalent as the following Lemma shows.

Lemma 4.1. *It holds that (\bar{Y}, \bar{Z}) is in $\mathcal{A}(F + x + \int_0^T \hat{\pi} \cdot d\hat{W}^{\hat{\theta}})$ if and only if (Y, Z) is in $\mathcal{A}^{\hat{\pi}}$ where $Y = \bar{Y} - X$ and $Z = \bar{Z} - \hat{\pi}$.*

Proof. Let b in \mathcal{D} , c in $bmo(\hat{\theta}) = bmo$ and a predictable such that $\int a M^{bc} dt$ is integrable and $g \geq by + c \cdot z - a$. We denote $C = \|F\|_{\infty}$. For (\bar{Y}, \bar{Z}) in $\mathcal{A}(F + x + \int \hat{\pi} \cdot d\hat{W}^{\hat{\theta}})$, since h is positive, it follows that $\bar{Y}_t \leq C + E[X_T | \mathcal{F}_t] = C + X_t + E[\int_t^T \hat{\pi} \cdot \hat{\theta} ds | \mathcal{F}_t]$. Since $M^{bc} \leq DM^c$ where D is a positive constant, it holds

$$M_t^{bc} Y_t = M^{bc}(\bar{Y}_t - X_t) \leq CDM_t^c + DM_t^c E\left[\int_t^T \hat{\pi} \cdot \hat{\theta} ds \middle| \mathcal{F}_t\right]$$

Let us show that $(M^{bc}Y)^+$ is integrable under $P^{\hat{\theta}}$. Let p be the reverse Hölder exponent of M^c under $P^{\hat{\theta}}$ and q its convex conjugate. Let \bar{p} be the reverse Hölder exponent of $M^{\hat{\theta}}$ under P and \bar{q} its convex conjugate. Using Hölder, Jensen as well as reverse Hölder inequalities, it holds

$$\begin{aligned} E^{\hat{\theta}} \left[M_t^c E \left[\int_t^T |\hat{\pi} \cdot \hat{\theta}| ds \middle| \mathcal{F}_t \right] \right] &\leq E^{\hat{\theta}} [(M_t^c)^p]^{1/p} E^{\hat{\theta}} \left[E \left[\int_0^T |\hat{\pi} \cdot \hat{\theta}| ds \middle| \mathcal{F}_t \right]^q \right]^{1/q} \\ &\leq C_p^{1/p} E \left[M_T^{\hat{\theta}} E \left[\int_0^T |\hat{\pi} \hat{\theta}| ds \middle| \mathcal{F}_t \right]^q \right]^{1/q} \leq C_p^{1/p} E \left[(M_T^{\hat{\theta}})^{\bar{p}} \right]^{1/(q\bar{p})} E \left[\left(\int_0^T |\hat{\pi} \hat{\theta}| ds \right)^{q\bar{q}} \right]^{1/(q\bar{q})} \\ &\leq C_p^{1/p} E \left[(M_T^{\hat{\theta}})^{\bar{p}} \right]^{1/(q\bar{p})} E \left[\left(\int_0^T \hat{\pi}^2 ds \right)^{q\bar{q}/2} \left(\int_0^T \hat{\theta}^2 ds \right)^{q\bar{q}/2} \right]^{1/(q\bar{q})} \\ &\leq C_p^{1/p} C_{\bar{p}}^{1/(q\bar{p})} E \left[\left(\int_0^T \hat{\pi}^2 ds \right)^{q\bar{q}} \right]^{1/(2q\bar{q})} E \left[\left(\int_0^T \hat{\theta}^2 ds \right)^{q\bar{q}} \right]^{1/(2q\bar{q})} \end{aligned}$$

Since $\hat{\theta}$ and $\hat{\pi}$ are in bmo it follows that both are in any $H^{2q\bar{q}}$, and therefore we deduce that

$$E^{\hat{\theta}} \left[(M_t^{bc} Y_t)^+ \right] < \infty$$

showing that $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$.

Reciprocally, let $(Y, Z) \in \mathcal{A}^{\hat{\pi}}$. Since $g(\hat{\pi}, X, y, z) \geq -(0, \hat{\theta}) \cdot z - \hat{\pi} \cdot \hat{\theta}$, it follows that $M^{-\hat{\theta}} Y^+$ is integrable under $P^{\hat{\theta}}$, that is Y^+ is integrable. However,

$$\int_0^t \bar{Z} dW \leq \bar{Y}_t - \bar{Y}_0 - \int_0^t h(\bar{Y}, \bar{Z}) ds \leq Y_t^+ - Y_0 + \int_0^t \hat{\pi} \cdot d\hat{W}^{\hat{\theta}}$$

Since Y^+ is integrable, it follows that

$$E \left[\left(\int_0^t \bar{Z} dW \right)^+ \right] \leq E[Y_t^+] - Y_0 + E \left[\left(\int_0^t \hat{\pi} \cdot d\hat{W}^{\hat{\theta}} \right)^+ \right] < \infty$$

Since $\hat{\pi}$ is in bmo and $\hat{\theta}$ is bmo showing therefore that (\bar{Y}, \bar{Z}) in $\mathcal{A}(F + x + \int \hat{\pi} \cdot d\hat{W}^{\hat{\theta}})$. \square

Example 4.2. In a simple case of complete financial market, that is for $n = d$ and $h(y, z) = \alpha y + z^2/(2\beta)$, where α and β are bounded processes such that $1/\beta$ is also a bounded process. Furthermore since $n = d$, we simplify the notations as usual setting $\theta := \hat{\theta}$, $W^\theta = \hat{W}^{\hat{\theta}}$ and $\pi = \hat{\pi}$. We further assume that θ as well as M_T^θ are bounded. It follows that

$$\partial_x g = \partial_y g = \alpha \quad \text{and} \quad \partial_r g = \partial_z g = \frac{z+r}{\beta} - \theta = v$$

According to the arguments yielding (2.5), in order to find an optimal solution to the optimization problem, it is sufficient to solve the following coupled forward backward stochastic differential equation

$$\begin{cases} \pi &= \beta(V + \theta) - Z \\ X_t &= x + \int_0^t (\beta(V + \theta) - Z) \cdot dW^\theta, \\ Y_t &= F - \int_t^T \left(\alpha Y + \alpha X + \frac{\beta}{2}(V^2 - \theta^2) \right) ds - \int_t^T Z \cdot dW^\theta, \\ U_t &= U_T + \int_t^T \frac{V^2}{2} ds + \int_t^T V \cdot dW^\theta, \\ U_T &= \int_0^T \alpha ds \end{cases}$$

with solution satisfying

- π is in bmo ;
- b is in \mathcal{D} .
- $\int g^*(c, b, b, c) M^{bc} dt$ is integrable.

One can easily deduce that the last backward stochastic differential equation

$$U_t = \int_0^T \alpha_s ds + \int_t^T \frac{V^2}{2} ds + \int_t^T V dW$$

admits a unique solution with V in bmo due to the assumption on α . Defining

$$X_T := \frac{1}{D_T^\alpha} \left(c + \int_0^T \frac{D^\alpha \beta(V^2 - \theta^2)}{2} dt + \int_0^T D^\alpha \beta(V + \theta) dW^\theta \right) - F,$$

it follows from the assumption on θ as well as the definition of V that X_T is bounded and we choose c such that $E^\theta[X_T] = x$.⁸ Thus, by martingale representation theorem, there exists a predictable process Γ in bmo such that

$$X_T = x + \int_0^T \Gamma \cdot dW^\theta.$$

⁸That is

$$c = \frac{1}{E^\theta[D_T^\alpha]} \left(x + E^\theta[F] - E^\theta \left[\frac{1}{D_T^\alpha} \left(\int_0^T \frac{D^\alpha \beta(V^2 - \theta^2)}{2} dt + \int_0^T D^\alpha \beta(V + \theta) \cdot dW^\theta \right) \right] \right).$$

Defining

$$\begin{cases} \pi^* &:= \Gamma \\ Z^* &:= \beta(V + \theta) - \Gamma \\ X^* &:= x + \int \pi^* \cdot dW^\theta \\ Y^* &:= \frac{1}{D^\alpha} \left(c + \int \frac{D^\alpha \beta(V^2 - \theta^2)}{2} ds + \int D^\alpha(Z + \Gamma) dW^\theta \right) - X^* \end{cases}$$

it follows that (X^*, Y^*, Z^*, V) is solution of the forward backward stochastic differential equation. We are left to check that this solution satisfies the integrability conditions. First, $\pi^* = \Gamma$ is in bmo . Second, $b = \alpha$ is bounded hence $b \in \mathcal{D}$. Finally, computation of the convex conjugate evaluated at (c, b, b, c) yields

$$g^*(c, \alpha, \alpha, c) = \frac{\beta(c + \theta)^2}{2} = \frac{\beta(V + \theta)^2}{2}$$

for which it clearly holds

$$\begin{aligned} E \left[\left| \int_0^t M^{bc} \frac{\beta(V + \theta)^2}{2} ds \right| \right] &\leq CE \left[\int_0^T M^V (V + \theta)^2 dt \right] \\ &\leq 2CE \left[\left| \langle M^V, \int V \cdot dW^\theta \rangle_T \right| \right] + 2C \|\theta\|_\infty^2 E \left[\int_0^T M^V dt \right] < \infty \end{aligned}$$

where $C \geq D^b \geq 0$. Hence, it follows that $\pi^* = \Gamma = \beta(V + \theta) - Z^*$ is an optimal solution to the optimization problem. It follows in particular that

$$\begin{aligned} \mathcal{E}_0 \left(F + x + \int_0^T \pi^* \cdot dW^\theta \right) &= Y_0^* + x = c \\ &= \frac{1}{E^\theta[D_T^{-\alpha}]} \left(x + E^\theta[F] - E^\theta \left[\frac{1}{D_T^\alpha} \left(\int_0^T \frac{D^\alpha \beta(V^2 - \theta^2)}{2D^\alpha} dt + \int_0^T D^\alpha \beta(V + \theta) \cdot dW^\theta \right) \right] \right) \quad (4.1) \end{aligned}$$

Example 4.3. In a similar way we can consider the previous example in the incomplete case, that is for $n < d$ and $h(y, z) = \alpha y + \hat{z}^2/(2\beta) + \tilde{z}^2/(2\beta)$, where α and β are bounded processes such that $1/\beta$ is also a bounded process. We further assume that α is deterministic and $\hat{\theta}$ as well as $M_T^{\hat{\theta}}$ are bounded. It follows that

$$\partial_x g = \partial_y g = \alpha \quad \text{and} \quad \partial_r g = \partial_z g = \frac{\hat{z} + r}{\beta} - \hat{\theta} = \hat{v}$$

Again, following the arguments yielding (2.5), in order to find an optimal solution to the optimization problem, it is sufficient to solve the following coupled forward backward stochastic

differential equation

$$\begin{cases} \hat{\pi} &= \beta(\hat{V} + \hat{\theta}) - \hat{Z} \\ \tilde{c} &= \partial_z g(\pi, X, Y, Z) = \frac{\tilde{Z}}{\beta} \\ X_t &= x + \int_0^t (\beta(\hat{V} + \hat{\theta}) - \hat{Z}) \cdot d\hat{W}^{\hat{\theta}}, \\ Y_t &= F - \int_t^T \left(\alpha Y + \alpha X + \frac{\beta}{2} (\hat{V}^2 - \hat{\theta}^2) + \frac{\tilde{Z}^2}{2\beta} \right) ds - \int_t^T Z \cdot dW^{\hat{\theta}}, \\ U_t &= U_T + \int_t^T \frac{\hat{V}^2 - \tilde{V}^2}{2} ds + \int_t^T V \cdot dW^{\hat{\theta}}, \\ U_T &= \int_0^T \left(\alpha + \frac{\tilde{c}^2}{2} \right) ds + \int_0^T \tilde{c} \cdot d\tilde{W} \end{cases}$$

with solution satisfying

- π is in bmo ;
- b is in d , and \tilde{c} is in bmo .
- $\int g^*(c, b, b, c) M^{bc} dt$ is integrable.

Assuming that \tilde{c} is in BMO, since α is deterministic, the last backward stochastic differential equation admits a unique solution with $V = (0, \tilde{c})$ in bmo . The following quadratic backward stochastic differential equation

$$\begin{cases} \Upsilon_t &= \Upsilon_T - \int_t^T \frac{\tilde{\Gamma}^2}{D^\alpha 2\beta} ds - \int_t^T \Gamma \cdot dW^{\hat{\theta}} \\ \Upsilon_T &= D_T^\alpha (F + x) + \int_0^T \frac{D^\alpha \beta \hat{\theta}^2}{2} dt - \int_0^T \beta \hat{\theta} (D^\alpha - D_T^\alpha) \cdot d\hat{W}^{\hat{\theta}} \end{cases}$$

admits a unique solution with Γ in bmo . It follows that the system is solved for

$$\begin{cases} \hat{\pi}^* &= \beta\theta - \hat{Z}^* = \beta\theta - \frac{\hat{\Gamma}}{D_T^\alpha} \\ \tilde{c} &= \frac{\tilde{Z}^*}{\beta} = \tilde{V} \\ \hat{Z}^* &= \beta\theta - \hat{\pi}^* = \frac{\hat{\Gamma}}{D_T^\alpha} \\ \tilde{Z}^* &= \frac{\tilde{\Gamma}}{D^\alpha} \\ \hat{V} &= 0 \\ X^* &= x + \int \hat{\pi}^* \cdot d\hat{W}^{\hat{\theta}} \\ Y^* &= \frac{1}{D^\alpha} \left(\Upsilon_0 - \int \frac{D^\alpha \beta \theta^2}{2} ds + \int \frac{\tilde{\Gamma}^2}{D^\alpha 2\beta} ds + \int D^\alpha \beta \theta \cdot d\hat{W}^{\hat{\theta}} + \int \tilde{\Gamma} \cdot d\tilde{W}^{\hat{\theta}} \right) - X^* \end{cases}$$

The fact that the integrability conditions of Theorem 2.3 are fulfilled follows the same argumentation as in Example 4.2. It follows in particular that

$$\mathcal{E}_0 \left(F + x + \int_0^T \hat{\pi}^* \cdot d\hat{W}^{\hat{\theta}} \right) = Y_0^* + x = \Upsilon_0 = D_T^\alpha x + E^{\hat{\theta}} \left[D_T^\alpha F + \int_0^T \frac{D^\alpha \beta \hat{\theta}^2}{2} dt - \int_0^T \frac{\tilde{\Gamma}^2}{D^\alpha 2\beta} dt \right] \quad (4.2) \quad \diamond$$

Cost of incompleteness With these two examples in mind, we can explicitly compare the cost of incompleteness when it comes to utility indifference pricing. To distinguish between the complete and incomplete market, we denote by π and $\hat{\pi}$ the set of d -dimensional and n -dimensional strategies, and define

$$x^* = \inf \left\{ x \in \mathbb{R} : \sup_{\pi \in bmo} \mathcal{E}_0 \left(F + x + \int_0^T \pi \cdot dW^\theta \right) > \mathcal{E}_0(F) \right\}$$

$$y^* = \inf \left\{ y \in \mathbb{R} : \sup_{\hat{\pi} \in \hat{bmo}} \mathcal{E}_0 \left(F + y + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta \right) > \mathcal{E}_0(F) \right\}$$

which represents the utility indifference prices for F in the complete and incomplete case, respectively. Intuitively, the price is higher in the incomplete case, which is the subject of the following proposition.

Proposition 4.4. *For every F in L^∞ such that $\mathcal{E}_0(F) < \infty$ it holds that $x^* \leq y^*$.*

Proof. We denote by I and J the set of those x and y in \mathbb{R} such that $\mathcal{E}_0(F + x + \int_0^T \pi \cdot dW^\theta) > \mathcal{E}_0(F)$ and $\mathcal{E}_0(F + y + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta) > \mathcal{E}_0(F)$ for some π in bmo and $\hat{\pi}$ in \hat{bmo} , respectively. Since \hat{bmo} is a subset of bmo , it follows that $I \subseteq J$ showing that $x^* = \inf I \leq \inf J = y^*$. \square

In what follows, if the supremum is attained, we denote by $\pi^*(x)$ and $\hat{\pi}^*(y)$ the optimal strategies such that

$$\mathcal{E}_0 \left(F + x + \int_0^T \pi^*(x) \cdot dW^\theta \right) = \max_{\pi \in bmo} \mathcal{E}_0 \left(F + x + \int_0^T \pi \cdot dW^\theta \right)$$

$$\mathcal{E}_0 \left(F + y + \int_0^T \hat{\pi}^*(y) \cdot d\hat{W}^\theta \right) = \max_{\hat{\pi} \in \hat{bmo}} \mathcal{E}_0 \left(F + y + \int_0^T \hat{\pi} \cdot d\hat{W}^\theta \right)$$

We consider our running example with driver $h(y, z) = \alpha y + z^2/(2\beta) = \alpha y + \hat{z}^2/(2\beta) + \tilde{z}^2/(2\beta)$ where α and β are bounded processes, β uniformly bounded away from 0. We further assume that the market price of risk θ in the complete case $n = d$, is uniformly bounded as well as M_T^θ . On the one hand, in to the computations of Example 4.2, since α is deterministic, the solution simplifies since V can be chosen equal to 0. Hence, according to (4.1), it follows that

$$\mathcal{E}_0 \left(F + x + \int_0^T \pi^*(x) \cdot dW^\theta \right) = D_T^\alpha x + E^\theta \left[D_T^\alpha F + \int_0^T \frac{D^\alpha \beta \theta^2}{2} dt \right]$$

On the other hand, according to (4.2) it holds

$$\mathcal{E}_0 \left(F + y + \int_0^T \hat{\pi}^*(y) \cdot d\hat{W}^\theta \right) = D_T^\alpha y + E^{\hat{\theta}} \left[D_T^\alpha F + \int_0^T \frac{D^\alpha \beta \hat{\theta}^2}{2} dt - \int_0^T \frac{\tilde{\Gamma}^2}{D^\alpha 2\beta} dt \right]$$

We deduce that

$$\begin{cases} x^* &= \frac{\mathcal{E}_0(F)}{D_T^\alpha} - E^\theta \left[F + \int_0^T \frac{D^\alpha \beta \theta^2}{2D_T^\alpha} dt \right] \\ y^* &= \frac{\mathcal{E}_0(F)}{D_T^\alpha} - E^{\hat{\theta}} \left[F + \int_0^T \frac{D^\alpha \beta \hat{\theta}^2}{2D_T^\alpha} dt - \int_0^T \frac{\tilde{\Gamma}^2}{D_T^\alpha D^\alpha 2\beta} dt \right] \end{cases}$$

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